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Mathematical Foundations of Classical Embeddings of Quantum Mechanical State Spaces

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In this paper we review the basic mathematical properties that allow the embedding of quantum state spaces into spaces of classical probability measures. In particular, the precise topological structures used for these immersions are described.

1. INTRODUCTION

The aim of this paper is to state in a precise and rigorous way the mathematical properties that allow the classical embeddings of quantum mechanical state spaces. By a classical embedding of a quantum mechanical state space we mean here the representation of quantum mechanical states as (classical) probability measures.

The motivation for such an investigation is due to some open problems in the mathematical and physical foundations of quantum mechanics: in particular, the quantum measurement problem suggests the possibility that some classical behavior of macroscopic systems could be understood in terms of quantum mechanics. In general, it is an open problem to understand the transition to the classical behavior of a quantum physical system. The theory reviewed in this paper can be a proper framework for these problems.

In this review we shall only deal with the mathematical aspects: we shall state precisely the definitions and their consequences; as a rule, the proofs will be omitted and referred to the existing literature.

The paper is organized as follows: in Section 2 we define the spaces of quantum states and classical (probability) measures and introduce their

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topologies. In Section 3 we describe the representation induced by a POV measure. In Section 4 we review the representation given by the Kadison map and its dual.

2. QUANTUM AND CLASSICAL STATE SPACES

To each quantum system a (complex, separable) Hilbert space H is associated; $B(H)$ denotes the space continuous operators on H, and $B_1(H)$ the subspace of trace-class operators on H [for the basic theory of operators in Hilbert spaces and, in particular, for the properties of trace-class operators we refer to the book of Reed and Simon (1980)]. The states of the quantum system are identified with positive trace-class operators of trace one; then $S(H)$, the set of states of the quantum system associated to the Hilbert space H, is

$$S(H) = \{T \in B_1(H) \colon T \geq 0, \ \mathrm{Tr}(T) = 1\}$$

Of course, the very definition of $S(H)$ suggests that it is natural to consider it as a subset of $B_1(H)$. Nevertheless, there is at least another useful and natural point of view. Each element of $S(H)$ defines a norm-bounded linear form on $B(H)$,

$$A \mapsto \mathrm{Tr}(TA), \qquad T \in S(H), \quad A \in B(H)$$

Hence $S(H)$ can be considered as a subset of the dual of $B(H)$ (viewed as a Banach space with respect to the operator norm). The former point of view is the one that will be used for the embedding considered in Section 3; the latter will be considered for the construction of the Kadison map in Section 4.

$B_1(H)$ is a Banach space with respect to the trace-norm, defined as

$$\|T\|_1 := \mathrm{Tr}(|T|), \qquad T \in B_1(H)$$

$S(H)$, as a subset of the Banach space $B_1(H)$, has the following properties:

1. $S(H)$ is a *convex set*; that is, if $0 < w < 1$ and $T_1, T_2 \in S(H)$, then

$$wT_1 + (1 - w)T_2 \in S(H)$$

2. The set of its *extreme points*, $\mathrm{Ex}(S(H))$, is

$$\mathrm{Ex}(S(H)) = \{T \in S(H) \colon T = P[x], \ x \in H\}$$

(where the operator $P[x]$ is defined by $P[x]y = (x, y)x$, $\forall y \in H$); in view of this characterization the elements of $\mathrm{Ex}(S(H))$ are called *vector states*, and $\mathrm{Ex}(S(H))$ will be denoted also by $V(H)$.

3. $S(H)$ is a *closed* subset of $B_1(H)$.

4. $S(H)$ is a *σ-convex* subset of $B_1(H)$ in the following precise sense: let $(w_i)_{i\geq 1}$ be a sequence of numbers such that $0 \leq w_i \leq 1, \forall i$, and $\sum_{i=1}^{\infty} w_i = 1$ (such a sequence will be called in the sequel a sequence of *weights*) and $(T_i)_{i\geq 1}$ be a sequence of elements of $S(H)$; then the series $\sum_{i=1}^{\infty} w_i T_i$ is convergent, in the sense of the trace-norm, to an element of $S(H)$.

5. From the spectral theorem for the compact operators it follows that

$$\mathrm{co}(V(H)) \subset \sigma - \mathrm{co}(V(H)) = \overline{\mathrm{co}(V(H))} = S(H) \tag{1}$$

[where $\mathrm{co}(V(H))$ denotes the convex hull of the set $V(H)$ and $\overline{\mathrm{co}(V(H))}$ the closed (with respect to the trace-norm) convex hull of $V(H)$].

Let us turn to the second possible point of view. $B(H)^*$ denotes the topological dual of the Banach space $B(H)$ (endowed with the operator norm); $B(H)^*$ is in a natural way a Banach space. We denote

$$S(B(H)) = \{\alpha \in B(H)^*: \alpha(A) \geq 0, \forall A \geq 0; \alpha(I) = 1\}$$

The elements of $S(B(H))$ are called *states on* $B(H)$. Obviously $S(B(H))$ is a convex subset of $B(H)^*$, and $\mathrm{Ex}(S(B(H)))$ denotes the set of its extreme points.

There is a natural inclusion of $S(H)$ in $B(H)^*$, and this inclusion is continuous [if $S(H)$ is endowed with the trace-norm]. Hence we have the topological immersion

$$S(H) \hookrightarrow B(H)^*$$

This means, in particular, that the σ-convex structure of $S(H)$ is preserved.

Various subsets of $S(H)$ can be characterized via this embedding:

1. $\mathrm{co}(V(H))$ is the subset of elements of $S(B(H))$ that are continuous with respect to the weak (equivalently, strong) operator topology of $B(H)$ (Takesaki, 1989).

2. $S(H) = \sigma - \mathrm{co}(V(H))$ is the subset of elements of $B(H)^*$ that are continuous with respect to the ultraweak (equivalently, ultrastrong) operator topology of $B(H)$ (Takesaki, 1989).

3. The closure of $\mathrm{co}(V(H))$ in the norm-topology of $B(H)^*$ [denoted $\overline{\mathrm{co}(V(H))}^{B(H)^*}$] is $S(H)$ itself; hence

$$\mathrm{co}(V(H)) \subset \sigma - \mathrm{co}(V(H)) = \overline{\mathrm{co}(V(H))}^{B(H)^*} = S(H) \tag{2}$$

[compare this with condition (1)].

$B(H)^*$ can be endowed with its weak-* topology, as dual of $B(H)$: this topology is defined by the separating family of seminorms on $B(H)^*$,

$$B(H) \ni \alpha \mapsto |\alpha(A)|, \qquad A \in B(H)$$

It is coarser than the norm one, and has the following properties:

1. The unit ball of $B(H)^*$ is compact in the weak-* topology (theorem of Banach–Alaoglu).
2. $S(B(H))$ is weak-* compact as a closed subset of a compact set.
3. The following density property holds:

$$\overline{\mathrm{co}(V(H))}^{\text{weak-*}} = S(B(H)) \tag{3}$$

[compare with conditions (1) and (2)]; a proof of this important result can be found in Emch (1972).

4. Moreover,

$$\mathrm{Ex}(S(B(H))) \subset \overline{V(H)}^{\text{weak-*}} \tag{4}$$

We shall use later this fundamental result in the construction of the Kadison map; for its proof see Emch (1972).

Ω denotes a compact (metrizable), topological space, $\mathscr{B}(\Omega)$ the σ-algebra of Borel sets of Ω, and $M(\Omega)$ the complex vector space of Borel complex measures on Ω (necessarily regular and bounded). $M(\Omega)$ is a Banach space with respect to the total variation norm

$$\|\mu\| := |\mu|(\Omega), \qquad \mu \in M(\Omega)$$

We recall that the total variation $|\mu|$ of an element μ of $M(\Omega)$ can be characterized as the smallest of the positive measures ν in $M(\Omega)$ for which

$$|\mu(X)| \leq \nu(X), \qquad \forall X \in \mathscr{B}(\Omega)$$

[for basic facts on measure theory on topological spaces we refer to Rudin (1973)].

$S(\Omega)$ denotes the subset of $M(\Omega)$ defined by

$$S(\Omega) = \{\mu \in M(\Omega) \colon \mu \geq 0, \mu(\Omega) = 1\}$$

Its elements of $S(\Omega)$ are the *classical states* (probability measures) on Ω. It is a convex subset of $M(\Omega)$; the set of its extreme elements, $\mathrm{Ex}(S(\Omega))$, coincides with the set of Dirac measures at the points of Ω. $S(\Omega)$ is σ-convex in the following precise sense: if $(w_i)_{i \geq 1}$ is a sequence of weights and $(\mu_i)_{i \geq 1}$ is a sequence of elements of $S(\Omega)$, then the series $\sum_{i=1}^{\infty} w_i \mu_i$ converges, in the norm of $M(\Omega)$, to an element of $S(\Omega)$.

$M(\Omega)$ can be identified with the dual of the Banach space $C(\Omega)$ of continuous functions on Ω with the sup norm; this identification is provided by the formula

$$\mu(f) = \int_\Omega f \, d\mu, \qquad \mu \in M(\Omega), \quad f \in C(\Omega)$$

We consider on $M(\Omega)$ its weak-* topology as dual of $C(\Omega)$; it is called the *vague topology* and it is defined by the separating family of seminorms

$$M(\Omega) \ni \mu \mapsto |\mu(f)|, \qquad f \in C(\Omega)$$

This topology is coarser than the norm one.

For $S(H)$ the following properties hold:

1. $S(\Omega)$ is vaguely closed in $M(\Omega)$; since the unit ball of $M(\Omega)$ is compact in the weak-* topology (Banach–Alaoglu theorem), $S(\Omega)$ is *compact*, as closed subset of a compact set.

2. The vague closure of $\mathrm{co}(\mathrm{Ex}(S(\Omega)))$ coincides with $S(\Omega)$ and we have the situation

$$\mathrm{co}(\mathrm{Ex}(S(\Omega))) \subset \sigma - \mathrm{co}(\mathrm{Ex}(S(\Omega))) \subset \overline{\mathrm{co}(\mathrm{Ex}(S(\Omega)))}^{\text{vague}} = S(\Omega)$$

[compare with conditions (1)–(3)].

3. EMBEDDING OF $S(H)$ VIA A POV MEASURE

Let Ω be a compact (metrizable) space; we fix a POV measure E on Ω; E is a mapping from $\mathscr{B}(\Omega)$ to the set of *positive* elements of $B(H)$ such that

1. $E(\varnothing) = 0$, $E(\Omega) = I$.
2. If $(X_i)_{i \geq 1}$ is a disjoint sequence of elements of $B(\Omega)$,

$$E\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} E(X_i)$$

[where the series on the right-hand side has to be understood in the weak (equivalently, strong) operator topology of $B(H)$].

Among POV measures we also consider PV measures, which are defined as those POV measures for which $E(X)$ is a projection for all $X \in \mathscr{B}(\Omega)$.

If $T \in B_1(H)$, we define

$$\mu_T(X) := \mathrm{Tr}(TE(X)), \qquad \forall X \in \mathscr{B}(\Omega)$$

Then $\mu_T \in M(\Omega)$; in fact, the trace has the property that, for any sequence $(A_i)_{i \geq 1}$ of elements of $B(H)$, converging weakly to $A \in B(H)$, $\mathrm{Tr}(TA) = \lim \mathrm{Tr}(TA_i)$; using this property and property (2) of the definition of a POV measure, one can easily prove that μ_T is indeed an element of $M(\Omega)$.

In this way, a map is defined

$$T \mapsto \mu_T(B_1(H) \to M(\Omega))$$

Its properties are:

1. It is linear.
2. It maps self-adjoint elements of $B_1(H)$ to real measures and positive elements of $B_1(H)$ to positive measures; moreover,

$$\mu_T \in S(\Omega) \qquad \text{if} \quad T \in S(H)$$

3. If $T \in B_1(H)$ is self-adjoint, then

$$|\mu_T| \leq \mu_{|T|}$$

In fact, μ_T is a real measure and we have the decompositions (Rudin, 1973)

$$\mu_T = \mu_T^+ - \mu_T^-, \qquad |\mu_T| = \mu_T^+ + \mu_T^-$$

where μ_T^+, μ_T^- are positive measures, minimal with respect to the possible decompositions

$$\mu_T = \nu_1 - \nu_2 \qquad (\nu_1, \nu_2 \text{ positive measures})$$

On the other hand, T can be decomposed as

$$T = T^+ - T^-, \qquad |T| = T^+ + T^-$$

where T^+, T^- are positive operators; then, by the linearity of $(T \mapsto \mu_T)$,

$$\mu_T = \mu_{T^+} - \mu_{T^-}$$

By the minimality property

$$\mu_T^+ \leq \mu_{T^+}, \qquad \mu_T^- \leq \mu_{T^-}$$

hence

$$|\mu_T| = \mu_T^+ + \mu_T^- \leq \mu_{T^+} + \mu_{T^-} = \mu_{T^+ + T^-} = \mu_{|T|}$$

4. For any $T \in B_1(H)$ we have the inequality

$$\|\mu_T\| \leq 2\|T\|_1$$

In fact, each $T \in B_1(H)$ can be decomposed as

$$T = \frac{T + T^*}{2} + i\frac{T - T^*}{2i} =: T_1 + iT_2$$

where T_1, T_2 are self-adjoint elements of $B_1(H)$; hence, by the previous item,

$$|\mu_T| \leq |\mu_{T_1}| + |\mu_{T_2}| \leq \mu_{|T_1|} + \mu_{|T_2|}$$

Then

$$\|\mu_T\| = |\mu_T|(\Omega) \le \mu_{|T_1|}(\Omega) + \mu_{|T_2|}(\Omega)$$

$$\mu_{|T_1|}(\Omega) = \left\|\frac{T+T^*}{2}\right\|_1 \le \frac{1}{2}(\|T\|_1 + \|T^*\|_1) = \|T\|_1$$

and similarly for $\mu_{|T_2|}$; we get in conclusion

$$\|\mu_T\| \le 2\|T\|_1$$

5. The previous inequality shows that the mapping $(T \mapsto \mu_T)$ is *continuous* from $B_1(H)$ to $M(\Omega)$, with the respective norms.

6. $(T \mapsto \mu_T)$ is *continuous* from $B_1(H)$ to $M(\Omega)$, with the vague topology.

7. $T \mapsto \mu_T(S(H) \to S(\Omega))$ preserves the σ-convex combinations.

8. Since $S(H)$ is a closed subset of $B_1(H)$ and $S(\Omega)$ is a closed subset of $M(\Omega)$ (with respect to the total variation norm and, hence, also to the vague topology), the restriction of $(T \mapsto \mu_T)$ to $S(H)$ is continuous from $S(H)$ to $S(\Omega)$, the latter considered either with the total variation norm or the vague topology.

In this way we have set up a continuous, σ-convex map between the quantum and classical state spaces. It turns out that this mapping has many interesting features: for example, the spectral properties of a self-adjoint operator whose spectral measure is E can be characterized in terms of the properties of the map $(T \mapsto \mu_T)$ that E induces (Cassinelli and Lahti, n.d.). Here we are interested only in the possibility of an embedding of $S(H)$ into $S(\Omega)$ via this map; hence we discuss only its *injectivity* and *surjectivity*; a complete discussion of its properties can be found in Bugajski *et al.* (n.d.).

The following properties hold for $(T \mapsto \mu_T)$:

1. If E is a PV measure, then $(T \mapsto \mu_T)$ is *never* injective; in fact, let $x \in H$ be any unit vector and $X \in \mathscr{B}(\Omega)$, and define

$$y := e^{iE(X)}x; \qquad T_1 := P[x], \qquad T_2 := P[y]$$

Then

$$\mu_{T_1} = \mu_{T_2}$$

but $T_1 \neq T_2$, unless x is an eigenvector of $E(X)$.

2. In case E is a PV measure, $(T \mapsto \mu_T)$ *can be* surjective; this is exactly when the spectrum of the self-adjoint operator corresponding to E by the spectral theorem contains only eigenvalues.

3. There are examples of POV measures such that the mappings $(T \mapsto \mu_T)$ they induce are *injective*; we refer for these examples to Bugajski *et al.* (n.d.) and references therein.

4. Exactly the POV measures that induce injective mappings *cannot* induce surjective mappings; hence injectivity and surjectivity exclude themselves mutually.

The case of injective mappings $(T \mapsto \mu_T)$ can be used to construct classical representations of quantum state spaces. The POV measures that induce injective mappings are sometimes called *informationally complete observables*.

4. THE KADISON MAP AND ITS DUAL

We denote by Ω the weak-* closure in $B(H)^*$ of the set $\mathrm{Ex}(S(B(H)))$:

$$\Omega = \overline{\mathrm{Ex}(S(B(H)))}^{\text{weak-*}}$$

Ω is compact as a closed subset of a compact set [the unit ball of $B(H)^*$]. The inclusion $V(H) \subset \Omega$ holds and also [compare (4)]

$$\mathrm{Ex}(S(B(H))) \subset \overline{V(H)}^{\text{weak-*}}$$

Hence

$$\overline{V(H)}^{\text{weak-*}} = \Omega$$

This shows that Ω is a natural compactification of $V(H)$; the topology of Ω [which is the restriction to Ω of the weak-* topology of $B(H)^*$] when restricted to $V(H)$ coincides with the restriction to $V(H)$ of the topology of the trace norm; then Ω is metrizable.

For all $A \in B(H)$ we define

$$f_A(\Omega \to \mathbf{C})$$

by

$$f_A(\omega) := \omega(A), \qquad \omega \in \Omega$$

f_A is a *continuous* function on Ω; in fact, for all $\omega_0 \in \Omega$ and each $\epsilon > 0$ the set

$$N_\epsilon(\omega_0) = \{\omega \in \Omega\colon |\omega(A) - \omega_0(A)| < \epsilon, \forall A \in B(H)\}$$

is a neighborhood of ω_0 (in the topology of Ω), if $\omega \in N_\epsilon(\omega_0)$, then

$$|\omega(A) - \omega_0(A)| < \epsilon$$

Hence

$$|f_A(\omega) - f_A(\omega_0)| < \epsilon$$

This shows that f_A is continuous.

We define a mapping

$$D(B(H) \to C(\Omega))$$

by

$$D(A) := f_A$$

The properties of D are the following

1. D is *linear.*
2. If A is self-adjoint, then D is *isometric*; in fact

$$\begin{aligned}
\|D(A)\|_\infty &= \sup\{|D(A)(\omega)|: \omega \in \Omega\} \\
&= \sup\{|\omega(A)|: \omega \in \Omega\} \\
&= \sup\{|\omega(A)|: \omega \in V(H)\} \\
&= \sup\{|(x, Ax)|: x \in H, \|x\| = 1\} \\
&= \|A\|
\end{aligned}$$

3. D is *continuous* from $B(H)$ to $C(\Omega)$; in fact, each element A of $B(H)$ can be decomposed as

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i} =: A_1 + iA_2$$

where A_1, A_2 are self-adjoint; then

$$\begin{aligned}
D(A) &= D(A_1) + iD(A_2) \\
\|D(A)\|_\infty &\le \|D(A_1)\|_\infty + \|D(A_2)\|_\infty \\
&= \|A_1\| + \|A_2\| \\
&\le 2\|A\|
\end{aligned}$$

4. D maps self-adjoint elements of $B(H)$ to real measures and positive elements to positive measures.

5. If $(A_n)_{n \ge 1}$ is an increasing, uniformly bounded sequence of elements of $B(H)$ that (necessarily) converges weakly to an element A of $B(H)$, then

$$D(A_n) \to D(A) \quad \text{in } C(\Omega) \quad \text{(that is, uniformly)}$$

In fact, from the weak convergence $A_n \to A$ we get that

$$D(A_n)(\omega) \to D(A)(\omega), \qquad \forall \omega \in V(H)$$

Since $V(H)$ is dense in Ω, it is easy to see that

$$D(A_n)(\omega) \to D(A)(\omega), \qquad \forall \omega \in \Omega$$

In conclusion, $D(A_n) \to D(A)$ pointwise, $(D(A_n))_{n \geq 1}$ is increasing in $C(\Omega)$, and $D(A)$ is continuous; hence $D(A_n) \to D(A)$ uniformly, by Dini's theorem.

6. From the previous point and the defining properties of a POV measure, we get that, if E is a POV measure on some Borel space $(\Gamma, \mathscr{B}(\Gamma))$, then $D \circ E$ is a $C_{\mathbf{R}}(\Omega)$-valued measure on the Borel space $(\Gamma, \mathscr{B}(\Gamma))$.

In the sequel we consider the restriction of D to the real Banach space $B_h(H)$ of self-adjoint elements of $B(H)$; denoting D (by a slight abuse of notation) this restriction, we have that

$$D(B_h(H) \to C_{\mathbf{R}}(\Omega))$$

is a positive isometry. Consider the dual D^* of this map:

$$D^*(C_{\mathbf{R}}(\Omega)^* \to B_h(H)^*)$$

Since Ω is a compact (metrizable) space, $C_{\mathbf{R}}(\Omega)^*$ is canonically identified with the real Banach space $M_{\mathbf{R}}(\Omega)$ of real Borel measures (necessarily regular and bounded) on Ω. The general properties of the dual of a linear map between Banach spaces assure that D^* has the following properties:

1. Since D is an isometry, D^* is *surjective* from $M_{\mathbf{R}}(\Omega)$ to $B_h(H)^*$ and the kernel of D^* is given by

$$\{\mu \in C_{\mathbf{R}}(\Omega) \colon \mu(f) = 0 \text{ for all } f \text{ in the range of } D\}$$

2. Since D^* is surjective, each $\omega \in B_h(H)^*$ can be written as

$$\omega = D^*(\mu)$$

where μ is some (not uniquely determined) Borel measure on Ω; in particular, each $P[x] \in V(H)$ can be written as $P[x] = D^*(\mu)$; then the definition of dual map implies that

$$\begin{aligned}(x, Ax) &= D^*(\mu(A)) \\ &= \mu(D(A)) \\ &= \int_{\Omega} D(A)(\omega)\, d\mu(\omega) \\ &= \int_{\Omega} f_A(\omega)\, d\mu(\omega)\end{aligned}$$

This formula provides a *classical representation* of the state $P[x]$: the mean value of each observable A in the state $P[x]$ is given by the integral of a (continuous) function over the "phase space" Ω.

3. Obviously, the measure μ, such that $\omega = D^*(\mu)$, is not uniquely determined, since D^* is not injective.

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